MATCHING WITH SINGLE-PEAKED PREFERENCES

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February 2018

Abstract. This paper studies two-sided one-to-one matching in a frictionless nontransferable utility model where agents are characterized by a univariate type and have single-peaked preferences characterized by an ideal type (decoupled from own type) and greater preference for matches closer to that type. Given some modest distributional assumptions, we recover a closed form for the matching function, and we prove the assignment is unique in finite markets. We also develop a generalization of the Sequential Preference Condition for uniqueness and show it applies to our model. Finally, we apply our results to a simple model of premarket educational investments and marriage matching.

JEL classification: C78, D15

Keywords: matching, preferences, marriage, uniqueness, pre-market investments.

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1. Introduction

In theoretical and empirical models of matching markets it is often useful to derive closed-form matching functions. These results can provide testable implications and intuition about sorting, and can make theoretical models with embedded matching problems tractable. Such results have been derived settings where agents have preferences over a single parameter which is either *vertical* (Becker (1973)), where all agents share a preference ordering over types; or *horizontal*, where agents prefer their own type ((Clark (2003), Clark (2007)). Closed forms have also been recovered in certain multidimensional settings (Lindenlaub (2017)).

In this paper we focus on a univariate one-to-one matching model with nontransferable utility (NTU) and generalize from vertical and horizonal preferences to single-peaked preferences, where each agent is characterized by both their own univariate trait and an ideal partner trait, preferring partners closer to that ideal. We recover a stable matching function that can be expressed in closed form, assuming that we have closed form continuous distributions of a continuum of agents. Applications include marriage and dating, as well as job search when bargaining over wages is difficult or impossible, as in many public sector and entry level professional jobs. In the discrete case with a finite set of agents, we can also show that the stable assignment is unique. For concreteness, we illustrate our results with a model of income/success and heterosexual marriage. Motivated by the stylized fact that many men prefer women of lower income or accomplishment than themselves (Bertrand, et al.), we assume some men have ideal partners whose success level is below their own–say, own success times $c < 1$, while others have vertical preferences and all women have vertical preferences. This represents a simple but nontrivial example of single-peaked preferences.

Finally, we apply these results to the literature on matching with premarket investments–specifically, we extend the marriage model described above by adding an educational investment stage before the matching stage, where success depends on effort and agents are differentiated by ability. Motivated by data suggesting gendered asymmetric preferences may be diminishing over time (Pew Research Center, 2011), we contrast this with a model where both men and women have vertical preferences, intended to represent a potential future where preferences are more symmetric.

We find that, under symmetric vertical preferences and symmetric male and female ability distributions, educational investments are efficient in the sense that two fundamental matching externalities cancel each other out, yielding invidual education choices that are identical to the surplus maximizing planner's solution. However, with traditional men in the market, high ability women are in surplus, since traditional men don't want to match to them. High ability women must therefore match assortatively to nontraditional men, and must accept a wider range of types, including lower ability men. Because this set of men is more dispersed and assignment is assortative, increasing one's education as a woman generally leads to a greater than one-to-one increase in partner education, incentivizing high type women to overinvest in order to attract the most educated nontraditional men. Conversely, high ability nontraditional men have an incentive to underinvest in education, since the set of women they match to has a much narrower range of abilities and thus a more compressed distribution of education. Traditional men, by contrast, are able to leverage their scarcity to match to their ideal partners regardless of the distribution of men and women, and thus their investment is invariant to the distributions of men and women so long as the conditions ensuring their scarcity are satisfied.

The results in this paper hinge on partially shared preference orderings. Eeckhout (2000) showed that both horizontal and vertical discrete matching problems have unique stable assignments because agents in both cases satisfy a weak notion of partially shared preference orderings–the Sequential Preference Condition (SPC). This condition simply requires that we can find an ordering where agents prefer their equal-ranked counterpart to any lower ranked partner. We show that, under some weak distributional assumptions, agents with single-peaked preferences also satisfy a limited notion of partially shared preferences. However, the SPC is too restrictive for our model, so we develop a generalized SPC. Essentially, we amend the SPC to allow agents to violate it so long as they satisfy an alternate notion of shared preferences. In the single-peaked case, this reduces to requiring that there not be too many agents who violate the SPC. This weaker condition also ensures unique stable assignments in finite markets. While single-peaked preferences constitute one application, the uniqueness result we derive applies to a large class of assignment problems.

This paper's contributions are relevant to the empirical and theoretical literature on matching. First, it makes analysis of models with embedded matching problems tractable in a single-peaked preference environment, as illustrated in the premarket investments application. Second, it contributes to the wide literature on *assortative* matching.[2] We show that stable assignments over single peaked preferences exhibits several different forms of assortation, which form testable predictions. In particular, individuals who are perfect matches–agent $m's$ ideal trait is agent $w's$ own trait and vice versa–will match stably, and they will exhibit what I term *converse positive assortative matching* (CPAM), where increases in an agent's ideal type correspond to increases in their match's own type. Individuals who don't get perfect matches, however, exhibit other forms of assortation. For agents who satisfy the SPC, we find *positive assortative matching* (PAM), where higher traits match to higher traits irrespective of ideal traits, or *negative assortative matching* (NAM), where higher traits match to lower traits, where the type of assortation depends on the relative orientation of the two distributions. We see PAM when agents generally prefer higher traits than their stable partner possesses (vertical preferences is one example of this). We see NAM when agents generally prefer lower traits than their stable partner possesses. However, agents who induce violations of the SPC can be seen as being in shortage, in a sense that will be developed later, and are able to leverage their scarcity to match to agents of their ideal type who do not find them ideal (CPAM for the side in shortage, PAM or NAM for the side in surplus).

This paper follows a rich literature on stable matching problems, starting with the seminal paper by Gale and Shapley (1964). Becker (1973) found that PAM occurs when there is a continuum of types and the utility of a match is increasing in types and nontransferable. Unlike Gale-Shapley, this requires no iterative process to find agent pairs in the stable matching, so it is suitable for use in theoretical models. However, it imposes the fairly onerous assumption of vertical preferences–higher types are universally preferred to lower types, and agents only care about one trait. Legros and Newman (2007) extended PAM and NAM results to a class of partially nontransferable utility problems, where there are limitations on the ability of some or all agents to transfer utility to their match.

Assuming horizontal preferences over a single trait where agents want to match to their own type, Clark (2003) gives an algorithm for finding stable matchings in a market with a finite set of agents. Clark (2007) then treats the univariate horizontal case with an infinite set of agents, finding a very simple matching result, which, like Becker's result for vertical preferences, is well suited to a theoretical model. Eeckhout (2000) develops a condition ensuring unique stable assignments that subsumes vertical and horizontal preferences. Clark (2006) also gives another such condition. Klumpp (2009) derives a very simple "inside-out" algorithm for horizontal matching with finitely many agents. Lindenlaub (2017) extends closed form matching functions to multivariate types.

[2]In one dimension, assortation is a matching structure where the type of an agent's match is monotonic in the agent's own type.

The remainder of this paper is organized as follows: Section 2 motivates the analysis with a simplified model of marriage matching, which elucidates most of the techniques later used in the more complex single-peaked case. Section 3 generalizes the model by allowing agents to have arbitrary single peaked preferences, and matching functions are derived given some additional assumptions, some of which we can relax. We also characterize a generalization of the SPC and prove that the most general single-peaked model we study satisfies it. Section 4 applies these results to the study of premarket educational investments and marriage. Section 5 concludes and an appendix follows.

2. Motivating Example: Marriage with Traditional Men

We'll start with an extremely simple model that captures the generality of single-peaked preferences over vertical preferences in a minimal setting. A stylized fact that has generally been neglected in the literature on marital matching is that many men in many cultures prefer partners who are less educated or successful than they are. This issue has been studied in other fields, and has recently been noted in other literatures in economics. Bertrand et al. (2013) shows that US marriages where the wife earns more than the husband fare worse along a variety of metrics–divorce rate, marital satisfaction, etc., and that, when one graphs the distribution of couples by wife's income share, there is a steep drop at one half–there are many marriages where wives earn just a bit less than their husbands, but much fewer where they earn a bit more.

At the same time, this gender asymmetry in preferences may be decreasing. A 2011 Pew study found that the younger the cohort, the more they claim to prefer more egalitarian marriages, with 72% of millennials prefering the statement "Husband and wife both have jobs/both take care of the house and children" to "Husband Provides/wife takes care of the house and children" as compared to just 56% of those 65+.

Therefore, we'll study a model of marriage matching over "success", which we can think of as income, education, status, etc. or some index thereof, with asymmetic preferences. That is, while all women and some men have vertical preferences, some proportion of men are *traditional,* prefering a partner with a finite success level that may be below their own.

2.1. Preliminaries.

We'll denote the sides W for women and M for men. The corresponding sets of agents are $\mathscr{W}$ and $\mathscr{M}$. We'll denote traditional men TM and nontraditional men NM. In this section, agents are characterized by success θ distributed [3] $F_M(\theta) = F_W(\theta) = F(\theta)$, where F is twice continuously differentiable with support Θ. The corresponding density is f. We'll denote an agent i with trait θ and preferred partner trait P in $\mathscr{W}$ as $w_{i\theta P}$, suppressing the index, trait, or preference as appropriate. We'll define $m_{i\theta P}$ symmetrically for $\mathscr{M}$. We'll denote the matching function that assigns each agent to their partner by μ. For each success level θ, a proportion $T(\theta) \in [0, 1]$ of men are tradititonal, while the remainder are nontraditional. Women and nontraditional men prefer higher success to lower success, while a traditional man m_θ has an ideal partner $P = c\theta - \delta$, where $\delta > 0$ and $0 < c < 1$.

We'll assume that agents face no search costs or other limitations to matching–that is, we'll focus on stable assignments. We'll also assume utility is nontransferable, so agents cannot bargain over the match surplus. An assignment is stable if there is no pair (m, w), which we may write as mw, such that $\mu(m) \underset{m}{\prec} w$ and $\mu(w) \underset{w}{\prec} m$.

For simplicity, we'll assume equal mass on each side of the market.

[3] For simplicity, we assume men and women share the same distribution.

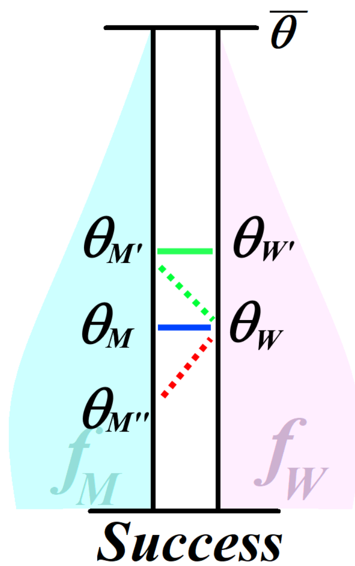

FIGURE 2.1. Potential blocking pairs under vertical preferences.

[4]In this paper we'll focus on a continuum case and normalize that mass to 1, affording us continuous densities and the ability to take derivatives and integrals, but will occasionally consider a finite analog, which is necessary to prove some uniqueness results.

- **Assumption 1 (MASS)** : Suppose an equal mass of agents on each side.

We'll also assume that every agent always accepts a match–that they never prefer to remain single. This assumption is more restrictive. If we relax this assumption, all the qualitative results we find go through, since any assignment problem failing this assumption coincides with one that satisfies it by retaining all matching agents and dropping those who remain single. However, it may be more difficult to recover closed forms for the matching function given a particular set of agents. This is because the decision to accept a match or not depends on the matching function, and the matching function may vary with agents' decisions to remain single.

- **Assumption 2 (DESP)** : every agent prefers any match to no match.

2.2. **Analysis.** We'll proceed informally in this section, as the results we derive are all special cases of propositions we'll formalize later. First, we'll briefly consider the case where $T(\theta) = 0$ for all $\theta \in \Theta$. This is simply the vertical matching setting studied by Becker (1973). In this setting, agents matching on percentile (PAM) is stable, and this assignment is unique in finite markets.

Consider this sketch of a stability proof: As shown in Figure 2.1, for any woman w with equal percentile partner m, matching to a partner below θ_m is unviable as a blocking pair since w prefers m, and matching to a partner with success above θ_m is unviable since m' prefers w'. Thus no blocking pair exists. Clearly, however, vertical preferences are not necessary for this result. We really only need that w prefers m to lower ranked men and m' prefers w' to lower ranked partners. This is precisely the SPC:

[4]If we relax this assumption, all subsequent results go through with the proviso that the extra mass of agents on the surplus side remains unmatched.

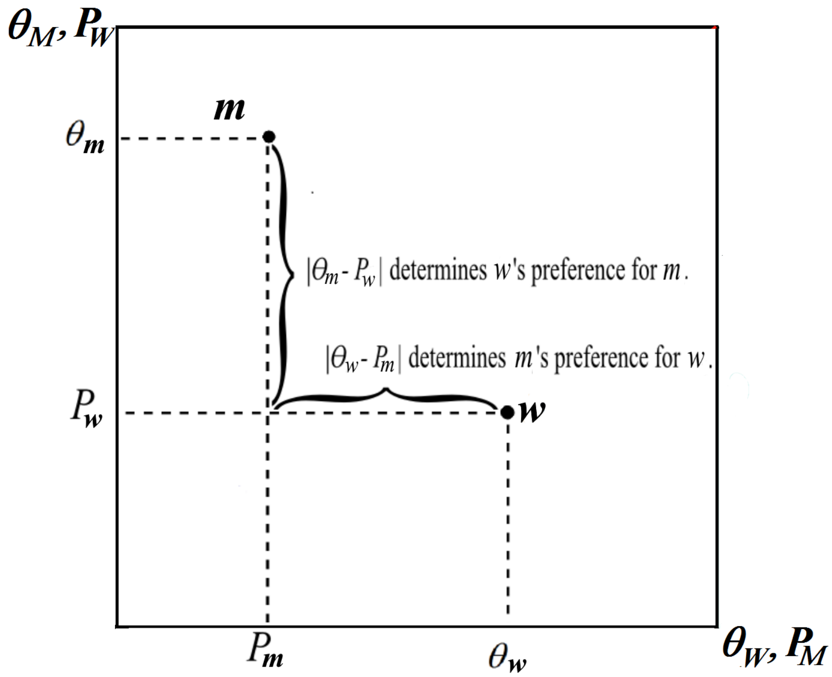

FIGURE 2.2. Comparing matches graphically using the overlay

Definition 1. (The Sequential Preference Condition): Given two finite ordered sets (w_i) and (m_i) and for each $j > i$, $m_i \underset{w_i}{\succ} m_j$ and $w_i \underset{m_i}{\succ} w_j$.

The SPC allows us to vastly weaken our assumptions on preferences while retaining the same elegant results as vertical preferences. This pursuit of more limited notions of shared preference orderings motivates the results of our paper. However, once we relax the assumption on T it becomes clear that the SPC is not general enough to accommodate even the most trivial examples of single-peaked preferences.

First, however, let's develop a framework to graph this problem. Since we're interested in single-peaked preferences characterized by type (θ, P), we'll use a bivariate distribution with density on the z-axis. We'll also overlay the distributions for men and women, transposing θ and P for one side of the market. As Figure 2.2 shows, this yields a simple interpretation for the relationship between any man and woman: vertical distance represents the woman's preference and horizontal distance represents the man's preference.

Now we're ready to graph the problem. We'll start with the vertical case, as in Figure 2.3. Because the axes are transposed for each side, the distribution of women is the reflection of the distribution of men along the 45° line. To make visualization easier, we interpret vertical preferences here as having an ideal type equal to the maximum type, rather than an unbounded preference for higher success. Since vertical distance represents women's disutility, we can see graphically that lower type men are less attractive, and similarly lower type women are less attractive. We can then find the stable assignment by starting with the highest ranked agents (top right) and matching outward along g, which represents the matching function. At each stage of the assignment, x matches to $y = g(x)$. We can analogize this process to unzipping a garment, with

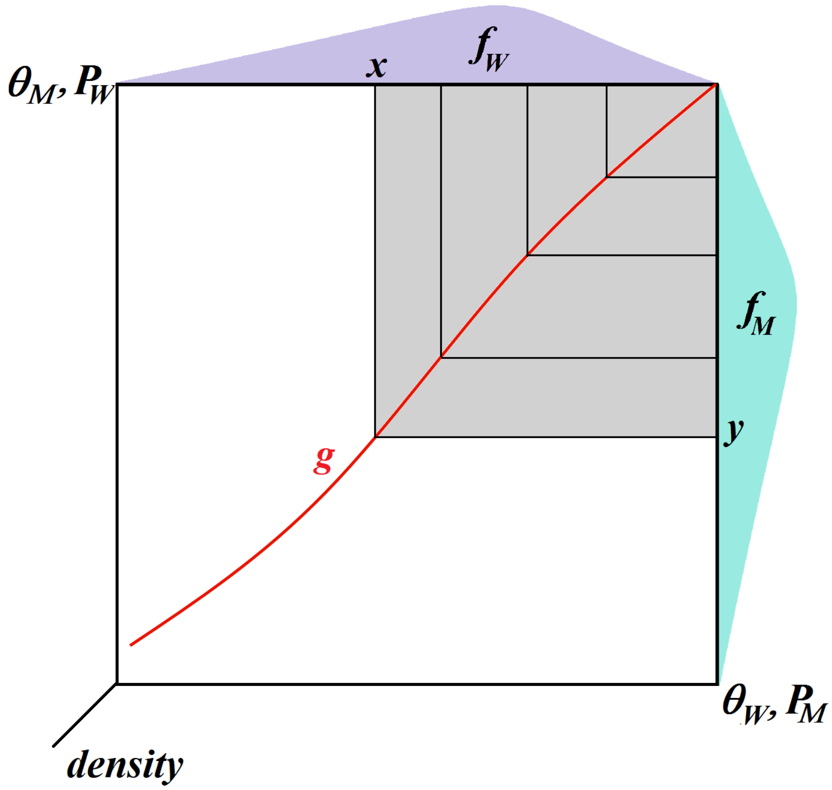

FIGURE 2.3. Matching with vertical preferences. The purple and teal densities are have one-dimensional support and extend outward along the z-axis. g corresponds to the matching function.

g representing the closed zipper and the edges going to x and y representing the unzipped halves. While all this setup is unnecessary to find stable assignments in the vertical case, it will turn out that more general single-peaked preferences can be analyzed in largely the same way.

Now we'll add traditional men (Figure 2.4). The matching process proceeds largely as before, with women and nontraditional men matching assortatively in own trait from the top right outward along g. We must now account for traditional men. We can see that, since the distribution of traditional men lies above [5]g, women must prefer a traditional man of their own stage to a nontraditional man in their own stage. That is, $m_{2b} \underset{w_2}{\succ} m_{2a}$. In fact, "unzipping" the market along g is still stable, and in finite markets uniquely so–traditional men find women in their stage ideal ($P_m = \theta_w$) so they are happy to match to them as well. It is simple to find a closed form for the matching function: we must have$\int_x^\infty f(\theta)d\theta = \int_{\frac{\theta+\delta}{c}}^\infty T(\theta)f(\theta)d\theta + \int_{g(x)}^\infty (1-T(\theta))f(\theta)d\theta$, and, assuming f and T are well behaved, we can easily solve for g. Intuitively, traditional men prefer lower success partners, which is advantageous to lower success women since it allows them to get a partner that is higher ranked than they are.

To ensure stability, however, we need to check for blocking pairs. Blocking pairs can't form within the set of nontradititonal men and women for the same reason they can't form with vertical preferences–better partners will reject you, and you'll reject worse partners. What about blocking pairs between traditional men and women? Women don't just prefer traditional men of their own stage to traditional men of later

[5]We can find condition on F, T, c, and δ to ensure this.

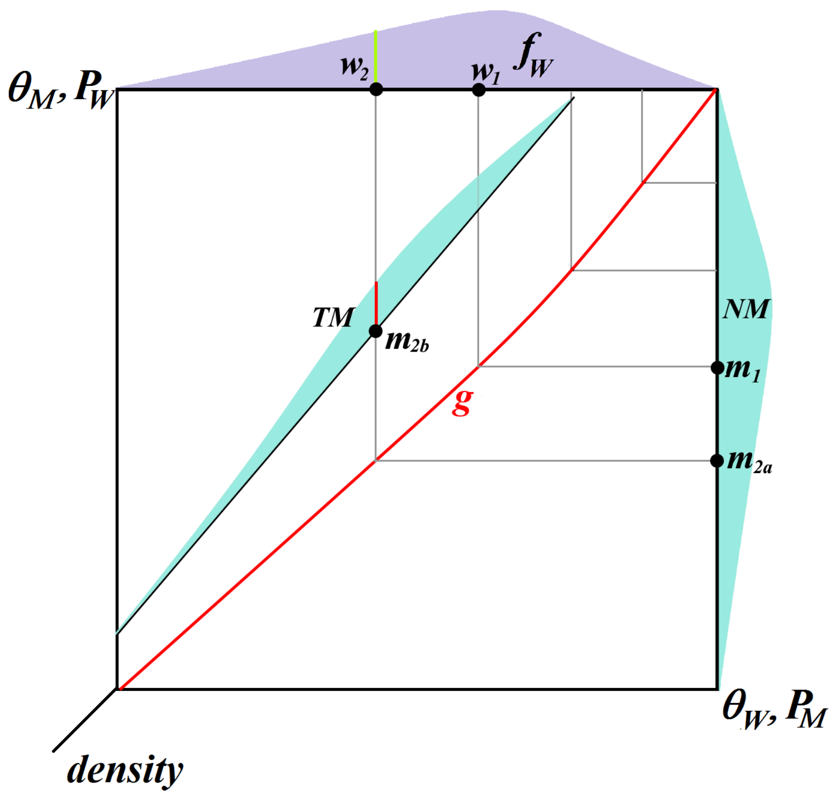

FIGURE 2.4. Matching with traditional men. The gray edges represent stages of the matching problem–each agent matches arbitrarily to another agent in their own stage.

stages, they may also prefer traditional men of later stages, as we can see in Figure 2.4,where w_1 prefers m_{2b} to m_1. This violates the SPC–taking stages of the assignment as rankings, the SPC requires that w_1 prefer m_1 to a lower ranked man like m_{2a}.

However, no blocking pair can form between w_1 and m_{2b} because m_{2b} is already getting his ideal match. The same analysis will hold for general single-peaked preferences–stability holds in spite of the SPC violation because the agents who violate it cannot form blocking pairs.

All of this assumes that there are enough women to match to traditional men. Thus, for these results to hold we must have $f(\theta)T(\theta) \leq f(c\theta - \delta)$ (these densities seen in red and green, respectively, in Figure 2.4) for all $\theta \in \Theta$, where $P = c\theta - \delta$ is the ideal partner success for a man of success θ. Summarizing, we've found that we can violate the SPC and still retain a unique, closed form matching function. However, we cannot violate it too much. This wiggle room will allow us to find analogous results for a wide array of single-peaked matching problems.

We'll see in Section 3.5 that we can think of this limited violation of the SPC as a special case of a weaker notion of partially shared preferences–when $f(\theta)T(\theta) \leq f(c\theta - \delta)$, those agents who violate the SPC satisfy an alternative notion of shared preferences analogous to an extremely weak form of a strict partial order.

3. GENERALIZATION TO ARBITRARY SINGLE PEAKED PREFERENCES

3.1. **Environment.** We now allow agent type and agent preference to vary independently, generalizing to arbitrary single peaked preferences. That is, for an agent w with preferred trait P, then for any $\theta_1, \theta_2 < P$, if $P - \theta_1 > P - \theta_2$ then $s_1 \underset{s}{\prec} s_2$, and for any $\theta_1, \theta_2 > P$, if $\theta_1 - P > \theta_2 - P$ then $s_1 \underset{s}{\prec} s_2$. . This

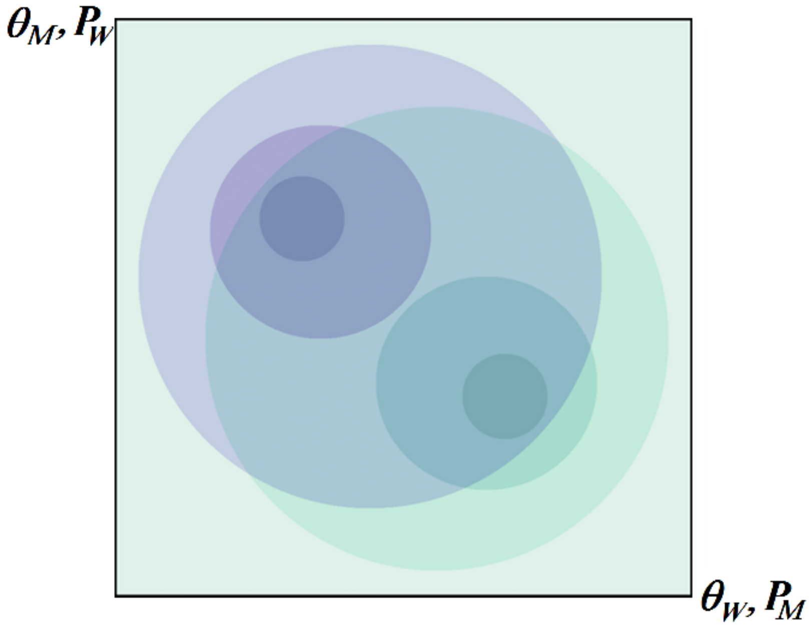

FIGURE 3.1. Overlaid contour plots of f_W (teal) and F_M (purple). Darker teal areas indicate a larger mass of women, and darker purple areas indicate a larger mass of men.

allows agents to prefer types other than their own. For example, men may prefer women of a different level of education than their own, or may prefer someone of a complementary disposition to make up for their shortcomings. Also, two individuals with the same characteristics may have different preferences over their match's characteristics, rather than e.g. a man's height uniquely determining his height preference.

It will be difficult to represent a z axis when distributions may have full support over trait and preference, so for this section we'll utilize contour plots, as seen in Figure 3.1. Darker purple represents higher density of women, and darker teal represents higher density off men.

3.2. **Analysis.** To find the stable assignment rule, we'll generalize the approach in Section 2. First, we have to deal with the overlapping support of the two distributions–something that doesn't appear in our previous setting. Luckily, this is simple–we simply match out overlapping agents to each other first, then proceed as before. Note that if m and w occupy the same point on the overlayed graphs, $m's$ ideal is $w's$ trait and vice versa. Thus, it must be stable to assign them to each other.

Lemma 2. *If, for each point (x, y), a measure of women with type (x, y) equal to $l_{xy} \equiv min\{f_W(x, y), f_M(y, x)\}$, match to a measure of men of type (y, x), none of these agents can form a blocking pair.*

Proof. There is a mass of at least $l_{\theta P}$ $w_{\theta P}$ agents and $m_{P\theta}$ agents. $w_{i\theta P}$ weakly prefers $m_{jP\theta}$ to any other man, and symmetrically $m_{iP\theta}$ prefers $w_{j\theta P}$ to any other woman. Then a mass $l_{\theta P}$ of $w_{\theta P}$ agents can be assigned to $m_{P\theta}$ agents stably. □

Having eliminated the overlap, we now need to construct densities for the remaining agents–remainder densities. Define

$$r_W(\theta, P) \equiv \max\{\frac{f_W(\theta, P) - f_M(P, \theta)}{\int_P \int_\theta (\max\{f_W(\theta, P) - f_M(P, \theta), 0\})\, d\theta dP}, 0\}$$

and

$$r_M(\theta, P) \equiv \max\{\frac{f_M(\theta, P) - f_W(P, \theta)}{\int_P \int_\theta (\max\{f_M(\theta, P) - f_W(P, \theta), 0\})\, d\theta dP}, 0\}$$

With R_W and R_M defined analogously. Define $r_M^T(P,\theta) \equiv r_M(\theta, P)$. Proceeding to the second stage, we'll keep MASS and DESP as before and add some additional distributional assumptions.

- **Assumption 3b** :
  - **3bi (SEPb)** the remainder densities r_W and r_M^T are separated by a curve $h : X \to Y$, where X represents both θ_W and P_M and Y represents both θ_M and P_W.
  - **3bii (MON)** $h(x)$ is monotonically increasing (decreasing) in x.

Figure 3.2 illustrates the remainder densities and an h that satisfies these conditions. Assumption 3b ensures that we'll be able to specify a limited notion of shared preference orderings, as before. This is a fairly easy assumption to satisfy, though it may be violated by multimodal distributions and distributions with widely varying tail weights.

At this point, we have a number of cases to consider. Figure 3.2 illustrates one, but h can be increasing or decreasing, and the remaining men can either be above h or below it. That yields four cases. In the interest of brevity, we'll restrict attention to just one case–the woman's problem when h is increasing and, for every man of type (θ, P), $P \geq h(\theta)$. The orientation of r_W and r_M and the choice of side to analyze are symmetric up to relabelling, and the slope of h only reverses the sign of the matching function, so we can proceed without loss of generality, with the proviso that we'll get PAM with an increasing h; and NAM with a decreasing h. We'll focus on the increasing case, noting the sign issue when relevant.

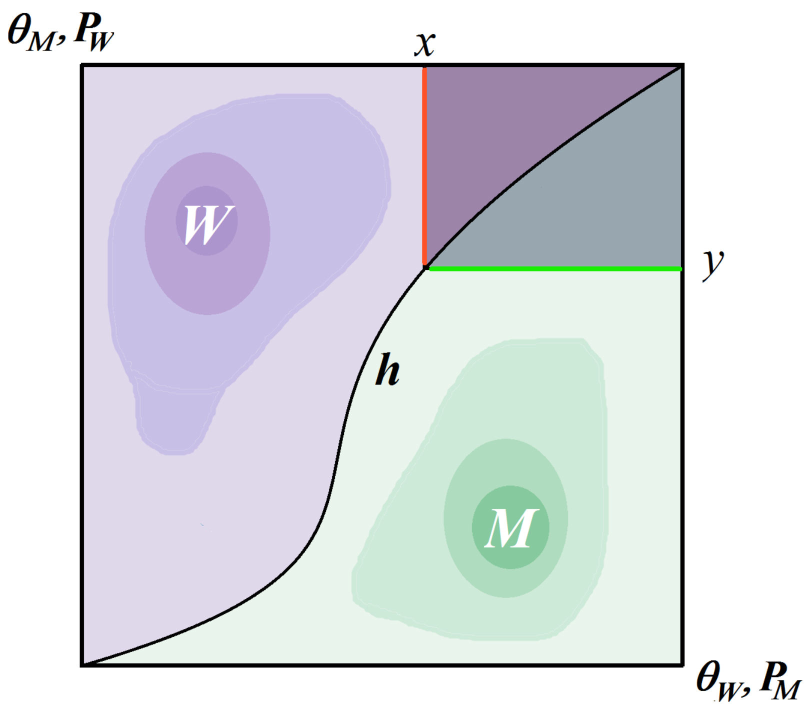

FIGURE 3.2. Iterative assignment from the upper right to the lower left on the remainder distributions of women and men. Men of trait x match to women of trait y.

3.3. **Benchmark Case: Symmetric Marginal Densities.** We now have two separated remainder densities, as in Figure 3.2. Before proving a more general case, let's sketch the solution to a simpler case where the SPC holds. We'll need an additional assumption, which implies the SPC in this setting.

- **Assumption 4a (MARGa)**: For each $(x, h(x))$, the marginal distributions $r_{M\theta}(x)$ and $r_{W\theta}(h(x))$ are equal.

We can proceed just as we did in Section 2, iteratively assigning agents from the top right and generating PAM in trait (NAM with a decreasing h). MARGa ensures that such an assignment is feasible. Visually, this resembles an "unzipping" of the market along the curve h, as mentioned before. Note that, while y trait women in Figure 3.4 don't agree on the ranking of every man, they agree on the ranking of all men who haven't yet been assigned, and agree that x trait men are most desirable (x trait men also find them the most attractive remaining women, ensuring stability.) This is exactly the SPC.

This only pins down the assignment in terms of trait, not ideal, so we'll make an additional assumption:

- **Assumption 5 (LEX)**: If $\theta_{w_i} = \theta_{w_j}$, $w_i \underset{m}{\prec} w_j$ if $P_{w_i} < P_{w_j} < \theta_m$ or $P_{w_i} > P_{w_j} > \theta_m$.

That is, we have lexicographic preferences, where, secondary to their partner's type, agents prefer matches to partners who like them more. This is not necessary to demonstrate the existence of any of the stable assignments discussed in this paper, but is necessary for establishing uniqueness. This assumption refines the set of stable assignments, requiring that, for a given x, x trait men match to $h(x)$ trait women in order of ideal type, with lower P women matching to lower P men. Since the SPC is satisfied in the equal marginals case this assignment is not only stable but unique up to identity and ideal trait in finite markets.

3.4. **General Case.** While the results sketched in 3.2 are extremely simple, MARGa is unlikely to be even approximately satisfied in a real world application. Having the marginal densities equal at any particular point on h is unlikely, much less at every point. To get a result with general application, we'll need to relax this assumption, as we did with traditional men in Section 2. As before, we'll define a function g corresponding to the matching functiton:

Definition 3. Define $g(x)$ such that, for $y = g(x)$,

1) when $\int_x^\infty r_{M\theta}(t)dt > \int_{h(x)}^\infty r_{W\theta}(t)dt$,

$$\int_x^\infty r_{M\theta}(t)dt = \int_x^\infty \int_y^\infty r_W(t,u)dtdu \tag{3.1}$$

2) and when $\int_x^\infty r_{M\theta}(t)dt < \int_{h(x)}^\infty r_{W\theta}(t)dt$,

$$\int_y^\infty \int_x^\infty r_M(t,u)dudt = \int_y^\infty r_{W\theta}(t)dt \tag{3.2}$$

Essentially, g is defined such that, for any x, the mass of women and men in the rectange with (x,g(x)) as the lower left vertex is equal, as seen in Figure 3.3. At any point in the matching process one side of the market (women in Figure 3.3) may be locally in shortage relative to the equal marginals case, in the sense that, if one tried to match agents along h, there wouldn't be enough shortage agents to meet the demand of the surplus side. Thus, the matching region must be extended outward into the shortage side, with (x,g(x)) falling in the shortage side. As the matching process proceeds along g the shortage may become a surplus, and g(x) is defined for both the men's surplus case and the women's surplus case, respectively.

The stable assignment will be, for each (x,g(x)), for the marginal women (comprising the surfaces α_1 and α_2 in Figure 3.3) to match to the marginal men (the surface β in Figure 3.5), with the assignment starting at (x,g(x)) and moving outward. Before proceeding, we need one assumption to replace MARGa:

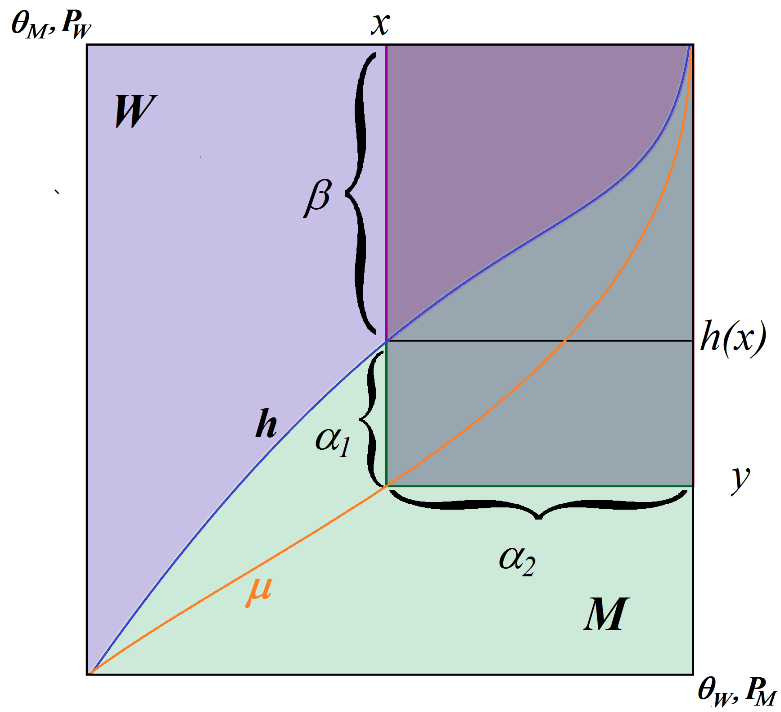

FIGURE 3.3. Generalized algorithm at a point where women are locally in surplus.

- **Assumption 4b (MARGb)**: if h is increasing, g is increasing. If h is decreasing, g is decreasing.

This ensures that assigning the marginal women (α_1 and α_2) to the marginal men (β) is feasible. Consider the case where h is increasing. If there is a point where g is decreasing, then following g would imply that previously assigned agents are unassigned at a later stage. Equivalently, if this condition fails, there is a point where the mass of agents in α_1 exceeds the mass in β.

Lemma 4. *MARGb implies*

i. if $\int_x^\infty r_{M\theta}(t)dt > \int_{h(x)}^\infty r_{W\theta}(t)dt$ *, then* $\int_y^{h(x)} r_W(t,x)dt \leq r_{M\theta}(x)$ *and*

ii. if $\int_x^\infty r_{M\theta}(t)dt < \int_{h(x)}^\infty r_{W\theta}(t)dt$*, then* $\int_y^{h(x)} r_W(t,x)dt \leq \int_{h(x)}^x r_M(x,t)dt$

and i. and ii. imply MARGb if $\int_x^\infty r_{M\theta}(t)dt > \int_{h(x)}^\infty r_{W\theta}(t)dt$ *implies* $\int_x^\infty r_W(g(x),u)du > 0$ *and* $\int_x^\infty r_{M\theta}(t)dt < \int_{h(x)}^\infty r_{W\theta}(t)dt$ *implies* $\int_{g(x)}^\infty r_M(t,x)dt > 0$.

Proof. When $\int_x^\infty r_{M\theta}(t)dt > \int_{h(x)}^\infty r_{W\theta}(t)dt$, we have $\int_x^\infty r_{M\theta}(t)dt = \int_x^\infty \int_{g(x)}^\infty r_W(t,u)dtdu$. Taking the derivative of both sides with respect to x, we have $r_{M\theta}(x) = \int_{g(x)}^\infty r_W(t,x)dt + \int_x^\infty r_W(g(x),u)g'(x)du = \int_y^{h(x)} r_W(t,x)dt + g'(x)\int_x^\infty r_W(g(x),u)du$. MARGb implies $g'(x) \geq 0$, so $r_{M\theta}(x) \geq \int_y^{h(x)} r_W(t,x)dt$. Similarly, $r_{M\theta}(x) \geq \int_y^{h(x)} r_W(t,x)dt$ and $\int_x^\infty r_W(g(x),u)du > 0$ and implies $g'(x) \geq 0$. A symmetric argument holds for case ii. □

Matching out agents in the rectangle defined by g means that, as men are matched out, women above g but below h must be assigned as well. When α_1 is smaller than β, the difference is made up by increasing the slope of g–putting more weight on α_2. However, if there are more α_1 agents than β agents, this assignment is infeasible no matter how low g' becomes. This condition is fairly easy to satisfy, but may be violated when the marginal densities are lesser for one side at the beginning of the second stage assignment process, generating a large gap between g and h, but greater for that same side later in the process, creating a large and dense α_1 surface.

Before stating the Lemma, let's define notation for the point on the surplus side where the matching function hits the kink on the shortage side. Define $P_M^*(\theta)$ implicitly by

$$\int_{h(\theta)}^{P_M^*(\theta)} r_M(\theta,u)du = \int_{g(\theta)}^{h(\theta)} r_W(t,\theta)dt \tag{3.3}$$

and $P_W^*(\theta)$ implicitly by

$$\int_{h^{-1}(\theta)}^{P_W^*(\theta)} r_W(\theta,u)du = \int_{g^{-1}(\theta)}^{h^{-1}(\theta)} r_M(t,\theta)dt \tag{3.4}$$

For surplus agents $m_{i\theta P}$ where $P \leq P_M^*(\theta)$, $\mu(m_{i\theta P}) = w_{j\theta_m\theta}$ where θ_m is defined implicitly by

$$\int_{h(\theta)}^{P} r_M(\theta,u)du = \int_{\theta_m}^{h(\theta)} r_W(t,\theta)dt \tag{3.5}$$

For surplus agents $m_{i\theta P}$ where $P > P_M^*(\theta)$, $\mu(m_{i\theta P}) = m_{jg(\theta)P_m}$ where P_m is defined implicitly by

$$\int_{h(\theta)}^{P} r_M(\theta,u)du = \int_{g(\theta)}^{h(\theta)} r_W(t,\theta)dt + \int_{\theta}^{P_m} r_W(g(\theta),u)du \tag{3.6}$$

For surplus agents $w_{i\theta P}$ where $P \leq P_W^*(\theta)$, $\mu(w_{i\theta P}) = m_{j\theta_m\theta}$ where θ_m is defined implicitly by

$$\int_{h^{-1}(\theta)}^{P} r_W(\theta,u)du = \int_{\theta_m}^{h^{-1}(\theta)} r_M(t,\theta)dt \tag{3.7}$$

For surplus agents $w_{i\theta P}$ where $P > P_W^*(\theta)$, $\mu(w_{i\theta P}) = m_{jg^{-1}(\theta)P_m}$ where P_m is defined implicitly by

$$\int_{h^{-1}(\theta)}^{P} r_M(\theta,u)du = \int_{g^{-1}(\theta)}^{h^{-1}(\theta)} r_W(t,\theta)dt + \int_{\theta}^{P_m} r_W(g^{-1}(\theta),u)du \tag{3.8}$$

Lemma 5. *Suppose assuptions 1-5 hold, h is increasing, and, for every M agent of type (θ, P), $P \geq h(\theta)$. If g(x)≤h(x), men of trait x and $P \leq P_M^*(x)$ match to women of trait θ_m satisfying Equation 3.5 and preference θ. men of trait x and $P > P_M^*(x)$ match to women of trait $g(\theta)$ and preference P_m satisfying Equation 3.6. If g(x)>h(x), women of trait x and $P \leq P_W^*(x)$ match to men of trait θ_m satisfying Equation 3.7 and preference θ. women of trait x and $P > P_W^*(x)$ match to men of trait $g^{-1}(\theta)$ and preference P_m satisfying Equation 3.8.*

Proof. The assignment satisfies feasibility by construction. Suppose a blocking pair $w_i m_j$ exists. Without loss, suppose the i pair matched before the j pair in the second stage assignment process. Also without loss, suppose m_i is in surplus. $w_i \underset{m_i}{\prec} w_j$ and $P_{m_i} > \theta_{w_j}$, so $\theta_{w_j} \geq \theta_{w_i}$. Suppose $\theta_{m_j} = \theta_{m_i}$. Then, because the remainder distributions are separated by h, $\theta_{w_i} < P_{m_i}$, and because i matched out first, $P_{m_i} < P_{m_j}$. Then $m_j \underset{w_i}{\prec} m_i$. Suppose not. Then, because $\theta_{m_j} < \theta_{m_i}$ and g is increasing, $P_{w_j} = \theta_{m_j}$, so $m_i \underset{w_j}{\prec} m_j$. Contradiction. ∎

We can now combine lemmas 2 and 5 to get the paper's main result. Define L as an assignment (unique up to identity and measure zero deviations) satisfying lemmas 1 and 3. Define the set of such assignments as $\mathscr{L}$.

Proposition 6. *(Single Peaked Two-sided Matching) Suppose assumptions 1-5 hold, h is increasing, and, for every M agent of type (θ, P), $P \geq h(\theta)$. Then any assignment L satisfying lemmas 2 and 5 is stable.*

Proof. The agents matched in lemma 2 are mutually ideal and cannot be a part of any blocking pair, and lemma 5 shows that the remaining agents cannot form blocking pairs. so both the first and second stage assignments are stable. □

If the distributions admit closed forms for the feasibility constraints, we can recover closed form matching functions. We can now characterize three regimes for the assignment and matching function.

(1) In the first stage, agents match to their mutually ideal partner, whose type is the converse of theirs. That is, $w_{\theta P}$ matches to $m_{P\theta}$. We'll call this *converse positive assortative matching*, or CPAM, because the assignment is (one-to-one) positively assortative, but across components of agent type. The higher your trait, the higher your partner's ideal, and the higher your ideal, the higher your partner's trait.
(2) In the second stage, some agents match via positive assortative matching (PAM) in trait if h is increasing, or negative assortative matching (NAM) in trait if h is decreasing. That is, if h is increasing, the higher your trait, the higher the trait of your partner; and if h is decreasing, the higher your trait, the lower the trait of your partner. In Figure 3.3 this corresponds to the agents in α_2 matching to β agents.
(3) In the second stage, some agents match via partial CPAM. In Figure 3.3 this corresponds to agents in α_1 matching to β agents. In fact, this regime admits a very elegant interpretation. Some of the agents who are locally in shortage (women in Figure 3.3) are able to leverage that scarcity to get matches that are ideal to them (partial CPAM, their ideal is their match's trait), but are not ideal to their partner. The surplus side, by contrast, faces matches that are generally quite distant from their ideal type.

Note that some of these regimes may not occur in a given assignment problem. We can provide an even more general result for one sided matching, as it is equivalent (with a continuum of agents or an even number of agents of each type) to a two sided matching problem where the sides are symmetric, automatically satisfying MARGa. Only regimes 1 and 2 will occur in this case.

Corollary 7. *(Single Peaked One-sided Matching) For any distribution $f(p, \theta)$, define $f_W(p, \theta) = F_M(p, \theta) = f(p, \theta)$. Suppose Assumptions 1-3 and 5 hold and, for every M agent of type (θ, P), $P \geq h(\theta)$. Then h(x)=x, $r_{M\theta}(x) = r_{W\theta}(x)$ for all x, and Proposition 4 holds.*

3.5. **Uniqueness.** We'll now establish uniqueness. The arguments used rely on exhaustion of a finite set of possible assignments, so cannot be directly generalized to the continuum case. Thus, we'll study the finite agent analogue to the environment discussed above. We'll also assume there is at most one of each type of agent–this is not necessary for the following proofs, but greatly simplifies the notation.

Lemma 8. *If a stable assignment $M \notin \mathscr{L}$ exists, then for any agent w_i (m_i) such that $m_i \neq \mu_M(w_i)$ $(w_i \neq \mu_M(m_i))$, $w_i\mu_M(w_i)$ $(\mu_M(m_i)m_i)$ must be an element of a cycle of M matches of the form $(w_{k_1}m_{k_2},\, w_{k_2}m_{k_3},\, \ldots,\, w_{k_N}m_{k_1})$ such that $k_j \neq k_l$ if $j \neq l$.*

Proof. Without loss, define $k_1 = i$ and consider an agent w_i. Then w_i must match to some m_{k_2}, $i \neq k_2$. Suppose that w_{k_j} matches to $m_{k_{j+1}}$ and $k_{j+1} \neq 1$ for all j<n, then $\mu_M(w_{k_n}) = m_{k_{n+1}} \neq m_{k_n}$. B is finite, so by exhaustion some w_{k_N} must match to m_{k_1}. □

- **Condition 1: Generalized Sequential Preference Condition (ORD)**
  - Consider two finite ordered sets (w_i) and (m_i). For any $j > i$, $m_i \underset{w_i}{\succ} m_j$ $(w_i \underset{m_i}{\succ} w_j)$, else

(1) **Pseudoasymmetry:** $w_j \underset{m_j}{\succ} w_i$ $(m_j \underset{w_j}{\succ} m_i)$, and

(2) **Pseudotransitivity:** For any $\{k_n\}_{n=1}^N$ such that $k_1 = i$, $m_{k_{n+1}} \underset{w_{k_n}}{\succ} m_{k_n}$ $(w_{k_{n+1}} \underset{m_{k_n}}{\succ} w_{k_n})$ for all n implies $m_{k_{N+1}} \underset{w_{k_N}}{\succ} m_i$ $(w_{k_{N+1}} \underset{m_{k_N}}{\succ} w_i)$.

We now return to the discussion of partially shared preference orderings. We noted that the SPC satisfied in both the horizontal and equal marginals cases. However, it may be violated in the more general case of Section 3.3. To account for this, we offer a weaker version of the SPC. Again, the SPC ensures that preferences are partially shared, with any two agents agreeing on the ordering of agents "less preferred" than either of them. To weaken the condition, we allow this form of shared preferences to be violated so long as another condition on shared preferences is satisfied, formalized by properties we term *pseudoasymmetry* and *pseudotransitivity*. Pseudoasymmetry ensures that, if an agent prefers a lower ranked partner to a partner of their own rank, that lower ranked partner must share the same preference ordering over i and j partners. Pseudotransitivity ensures that, if we can find a sequence of agents on one side of the market that each prefer the next agent's partner to their own, these preferences are transitive in the sense that the agent at the end of the sequence agrees with the composition of pairwise rankings up to that point and prefers their partner to the first partner in the sequence, extending transitivity beyond one individual's pairwise comparisions to a very limited form of consensus between agents. The intuition behind the guarantee of uniqueness is this: consider any alternative assignment M. These properties mean that, if $w_i m_i$ and $w_j m_j$ violate the SPC, by, say, the higher ranked w_i prefering and matching to the lower ranked m_j under M, it must be because m_j is more attractive than m_i in a way that w_j agrees on. Then, since m_j prefers their partner to w_i, it must be that w_j has an even more attractive partner m_k. This creates another agent who needs to receive an even more attractive partner, and so on. Because of the limited notion of shared preference orderings, no matter how far this chain of alternate assignments extends, the w agent at the end shares $w_i's$ distaste for m_i relative to later partners, and, when the chain inevitably closes into a cycle by the finiteness of the set of agents, we find a contradiction since the last woman must now prefer the next man, who is m_i.

Proposition 9. *Given that ORD is satisfied after mutually ideal assignments have been removed from consideration, there exists a unique stable matching.*

Proof. See Appendix A. □

Proposition 10. *Given assumptions 1-5, if there is a finite set of agents, h is increasing, preferences are strict, and, for every remainder M agent of type (θ, P), $P \geq h(\theta)$, preferences satisfy ORD after mutually ideal assignments have been removed from consideration*

Proof. See Appendix A. □

Corollary 11. *Given assumptions 1-5, if there is a finite set of agents, h is increasing, preferences are strict, and, for every remainder M agent of type (θ, P), $P \geq h(\theta)$, there exists a unique stable assignment.*

4. Application: Premarital Education Investments

4.1. **Motivation.** The results of Section 3 are fairly abstract, so we'll return to the case from Section 2 and illustrate how it can be embedded in a larger economic model. This will also contribute to the literature on matching markets with premarket investments. Concretely, we'll study a two period model where men and women invest in education in the first period and marry, work, and receive a payoff (success) in the second. Their payoff depends on their education level both directly through their success, and indirectly through their marriage assignment–where matching occurs over Success, as before.[6] Thus, agents will take matching outcomes into consideration when choosing their investment.

While the previous literature (Iyigun and Walsh (2007), Chiappori, Iyigun, and Weiss (2009)) assumes vertical preferences, we'll leverage the results of Section 3 to study an environment with more general single peaked preferences. In particular, we'll be able to analyze a marraige market that encorporates a stylized fact that has been thusfar neglected in the literature–many men prefer partners who are less educated or successful than they are. This issue has been studied in other fields, and has recently been noted in economics. Bertrand (2013) shows that marriages where the wife earns more than the husband fare worse along a variety of metrics–divorce rate, marital satisfaction, etc., and that, when one graphs the distribution of couples by wife's income share, there is a steep drop at 0.5–there are many marriages where wives earn just a bit less than their husbands, but much fewer where they earn a bit more.

At the same time, this gender asymmetry in preferences may be decreasing. A 2011 Pew study found that the younger the cohort, the more they claim to prefer more egalitarian marriages, with 72% of millennials prefering the statement "Husband and wife both have jobs/both take care of the house and children" to "Husband Provides/wife takes care of the house and children" as compared to just 56% of those 65+.

Therefore, we'll study a model with asymmetic preferences, where some proportion of men are *traditional*, with an ideal trait below their own, while other men and all women have vertical preferences, preferring higher achieving partners. We'll compare this to a hypothetical symmetric case where all men and all women have vertical preferences to see how educational investments may change as preferences change. In particular, we'll focus on matching externalities, and how these lead individuals to over or under-invest in education relative to a planner's first-best allocation of educational investment and matching assignments.

4.2. **Preliminaries.** We'll utilize the same setting as Section 2, with a few changes. When considering the case with traditional men, we'll denote the three groups via $\Gamma \in \{W, TM, NM\}$. Agents are now characterized by ability a distributed $H_M(a) = H_W(a) = H(a)$, where H is twice continuously differentiable and has bounded support $[\underline{a}, \overline{a}]$, $0 < \underline{a} < \overline{a}$. The corresponding density is h. Suppose $\xi < h(a) < C$ for all a in $[\underline{a}, \overline{a}]$.

In the first period, agents invest in educational effort $e_i \in [a, a + \overline{e})$[7] at cost $\phi(e_i - a)$, where $\phi(0) = 0$, $\phi'(0) = 0$, $\phi' \geq 0$, $\phi'' > 0$, and $lim_{e->a+\overline{e}}\phi[e] = \infty$. An agent's expected "quality" or success θ_i depends on this investment: $E[\theta_i] = e_i$. As a matching problem with an initial investment stage, this model may feature coordination failure equilibria, where no one invests above or below a cutoff because they know the other side will not either. To eliminate these equilibria and ensure the utility function is differentiable, we'll add a

[6] In this model, education and income/success will be largely interchangeable.

[7] To simplify the analysis, we'll assume there's a limited range of education level possible for each agent, depending on their ability. This allows us to easily exclude equilibria where investment decisions are paradoxical.

mean zero noise component ϵ to quality, with support $(-\bar{\epsilon}, \bar{\epsilon})$, $\bar{\epsilon} > 0$, and a thrice continuously differentiable distribution Q with density q, where q and q' are Lipschitz continuous, $q(-x) = q(x)$, $q(|x|) \leq q(|y|)$ if $|y| \leq |x|$, and $Q(\bar{\epsilon}/2) - Q(-\bar{\epsilon}/2) \leq 1/2$ for $x, y \in (-\bar{\epsilon}, \bar{\epsilon})$. Thus $\theta_i = e_i + \epsilon_i$. This yields success distributions $F_W(\theta)$, $F_{NM}(\theta)$, and $F_{TM}(\theta)$ with corresponding densities f_W, f_{NM}, and f_{TM}. Agents match based on their own success and ideal partner success, as described in Section 3. The proportion of traditional men is $0 < \underline{T} < T(a) < \overline{T} < 1$ for all a. The utility functions are as follows–we'll denote ex-post utility for agent i in group Γ as $u_{\Gamma i}(e_i, \theta_i, \mu_i)$, where $\mu_i \equiv \mu(\gamma_i)$. When unambiguous, we'll suppress μ, yielding $u_{\Gamma i}(e_i, \theta_i)$.

We'll now define the random variable η as an analogue to μ that instead maps from an agent's educatiton to the distribution of success levels of their partner: $\eta_\Gamma(e)$. We'll show that e is a well defined function of a, so we'll often write this as $\eta_\Gamma(a)$. $\eta_{p\Gamma}(\theta)$ gives match success for own success θ.

The ex-ante lifetime utlity for women who match is

$$U_{Wi}(e_i, a_i) \equiv Eu_{Wi}(e_i, \theta_i) = \frac{e_i + E[\eta_W(e_i)]}{2} - \phi(e_i - a_i). \tag{4.1}$$

For nontraditional men who match, ex-ante lifetime utility is

$$U_{NMi}(e_i, a_i) \equiv Eu_{NMi}(e_i, \theta_i) = \frac{e_i + E[\eta_{NM}(e_i)]}{2} - \phi(e_i - a_i). \tag{4.2}$$

For traditional men who match, ex-ante lifetime utility is

$$U_{TMi}(e_i, a_i) \equiv Eu_{TMi}(e_i, \theta_i) = \frac{e_i + E[\eta_{NM}(e_i)]}{2} - \phi(e_i - a_i) + z - E[\psi(\eta_{TM}(e_i) - c(e_i + \epsilon_i) + \delta)] \tag{4.3}$$

Later, we'll briefly consider the case where we relax DESP. For women, ex-ante utility for remaining unmatched is

$$U_{Wi}(e_i, a_i) \equiv e_i - \phi(e_i - a_i) - v. \tag{4.4}$$

For all men, ex-ante utility for remaining unmatched is

$$U_{Mi}(e_i, a_i) \equiv e_i - \phi(e_i - a_i) - v. \tag{4.5}$$

Where $\psi'(x){>}0$ if $x > 0$, $\psi''(x) > 0$, $\psi'(0) = 0$, $\psi(x) = \psi(-x)$, and $\psi(0) = 0$. $z \in \mathbb{R}^+$ is a constant that will be relevant when we consider outside options–without it, traditional men would always have relatively better outside options than nontraditional men. Similarly, $v \in \mathbb{R}^+$ represents the disutility of remaining unmatched, creating an incentive to find a partner.

We can interpret the utility function as follows: agents generally prefer matching to not (v is positive), and benefit from additional educational effort via greater education, income, or other benefits, indexed by success θ. These benefits are not wholly determined by effort, but depend on underlying ability and are subject to some small degree of uncertainty. Agents assess partnerships based on the success of their partner. If they marry, partners split the match surplus evenly, and cannot bargain over it. This can be motivated by the difficulty of credibly precommitting to a particular apportionment of surplus in a relationship, the infrequency of commitments like prenuptual agreements, and legal precedents for the outside option of divorce that militate towards an even split of assets. For traditional men, they also experience disutility from additional partner education above a certain point. This counteracts the direct positive effect of partner education on utility, and, given the restrictions placed on ψ, a traditional man will prefer a partner with limited success to one arbitrarily successful. This specification is fairly general, with the main restriction being that, because we sum own and partner education to get individual utility, we are imposing a strong additive seperability assumption. While it would be interesting to study a more general model, additive seperability dramatically simplifies the analysis, so for the sake of parsimony we maintain that assumption.

This specification reveals two externalities that may drive a wedge between the incentives of individuals and those of a social planner, and will thus underly our analysis of investment efficiency. The first externality–the rivalry externality–derives from the fact that each agent can only match to one partner, so, for example, one woman's gain is another woman's loss. In particular, if a woman increases her effort, her partner's expected success increases, increasing her utility. However[8], when some other woman is forced out of her previous match. If preferences are vertical, the other woman's loss is exactly equal to the first woman's gain. Conversely, agents impose positive externalities on the other side of the market via the NTU externality. For example, if a man increases e, that raises the average education level for men, giving a better average assignment to women. Because of NTU, women can't compensate men (and vice versa) for the benefits their investment decisions provide, creating an externality.

Note that we're looking at a case with a continuum of agents, while our uniqueness result is for finite markets. Additionally, because we've added an investment stage to the problem, Uniqueness may not hold even in a finite analogue. We'll proceed with the proviso that the equilibria we find may not be unique.

4.3. **Vertical Preferences.** We'll now consider the case of vertical preferences. Since $T(a) = 0$ for all a, we can drop traditional men and simply identify distributions and utilities of men by the subscript M. Since the ability distributions and preferences are symmetric, we'll look for a symmetric equilibrium.

$$\int_{\mathscr{W}} U_{Wi}(e_i, a_i) + \int_{\mathscr{M}} U_{Mj}(e_j, a_j) \tag{4.6}$$

subject to assignment feasibility. We'll show that this equivalent to maximizing $e_i - \phi(e_i - a_i)$ for each agent independently.

Lemma 12. *Given vertical preferences, maximizing $e_i - \phi(e_i - a_i)$ for each agent is sufficient to maximize total market surplus, independent of assignment.*

Proof.

$$\begin{aligned}
&\int_{\mathscr{W}} Eu_{Wi}(e_i, \theta_i) + \int_{\mathscr{M}} Eu_{Mj}(e_j, \theta_i) \\
&= \int_{\mathscr{W}} \left(\frac{e_i + E[\eta_W(e_i)]}{2} - \phi(e_i - a_i)\right) + \int_{\mathscr{M}} \left(\frac{e_j + E[\eta_M(e_j)]}{2} - \phi(e_j - a_j)\right) \\
&= \int_{\mathscr{W}} \left(\frac{e_i}{2} - \phi(e_i - a_i)\right) + \int_{\mathscr{W}} (E[\eta_W(e_i)]/2) + \int_{\mathscr{M}} \left(\frac{e_j}{2} - \phi(e_j - a_j)\right) + \int_{\mathscr{M}} (E[\eta_M(e_j)]/2)
\end{aligned}$$

But by feasibility, $\int_{\mathscr{W}} E[\eta_W(e_i)] = \int_{\mathscr{M}} e_j$ and $\int_{\mathscr{M}} E[\eta_M(e_j)] = \int_{\mathscr{W}} e_i$, so we have

$$= \int_{\mathscr{W}} e_i - \phi(e_i - a_i) + \int_{\mathscr{M}} e_j - \phi(e_j - a_j).$$

That is, total surplus is independent of assignment. Therefore, to maximize surplus, we can simply maximize $e_i - \phi(e_i - a_i)$ for each agent. □

[8]This example elides the continuum of agents and uncertainty of assignment for the sake of expositional clarity–these complications do not meaninfully change the externality, but make its interpretation slightly more complex.

This yields a first order condition of $e_i = \phi_x^{-1}(1) + a_i$. We'll show that this is precisely the investment level individuals make. This implies that the decentralized equilibrium yields the first best level of investment for all agents.

Proposition 13. *Given vertical preferences, every agent investing in the first best education level* $e_i = \phi_x^{-1}(1) + a_i$ *is a Nash Equilibrium.*

Proof. $H_W = H_M$ and $e_i = \phi_x^{-1}(1) + a_i$ for all i implies $F_M = F_W$. Assigning every agent to a partner of identical θ must be stable by Proposition 4. Then $\frac{\partial}{\partial e_i} U_{Si}(e_i, a_i) = \frac{1+1}{2} - \phi_x(e_i - a_i)$ and $e_i^* = \phi_x^{-1}(1) + a_i$, rationalizing that investment. By Lemma 10, total surplus is maximized when $\phi_x^{-1}(1) + a_i. = 0$, so $\mathbf{e}^*$ maximizes total surplus. $\square$

Note that this result depends on the symmetry of H_W and H_M. Symmetry ensures that perfect assortation is possible, making the marginal increase in utility from one's match 1/2. This corresponds to the rivalry externality. Conversely, the benefit an agent recieves from their partner increasing their type–the NTU externality–is 1/2 as well. Thus, under symmetry, the positive and negative externalities cancel out, giving agents the same incentives as the planner.

4.4. **Adding Traditional Men.** Now we can address the case with traditional men. The ideal partner education for a traditional man with education e_i and noise parameter ϵ is is determined by the following first order condition:

$$\frac{\partial}{\partial \eta_i} u_{TMi}(e_i, \theta_i) = \frac{1}{2} - \psi_x(\eta_{pTM} - c\theta_i + \delta) = 0$$

$$\eta^*_{pTM} = c\theta_i + \psi_x^{-1}(1/2) - \delta \tag{4.7}$$

Note that 4.7 implies that the traditional man who matches with a woman of success level θ will have success level $\frac{\theta - \psi_x^{-1}(1/2) + \delta}{c}$. We'll focus on the case where traditional men are in shortage as described in the preceding sections. Thus, we need to establish two results:

Lemma 14. *We can choose parameters T, $\bar{e}, \bar{\epsilon}$ such that:*

i) the distribution of traditional men falls above the curve g representing the matching function–traditional men are in shortage.

ii) g is monotonically increasing–MARGb is satisfied.

Proof. Appendix B. $\square$

Given the above, we have

$$U_{TMi}(e_i, a_i) = \frac{e_i + E[c\theta_i + \psi_x^{-1}(1/2)]}{2} - \phi(e_i - a_i) + z - E[\psi((c\theta_i + \psi_x^{-1}(1/2)) - c(e_i + \epsilon_i))]$$

$$U_{TMi}(e_i, a_i) = \frac{e_i(1+c) + \psi_x^{-1}(1/2)}{2} - \phi(e_i - a_i) + z - \psi(\psi_x^{-1}(1/2)) \tag{4.8}$$

Which yields an optimal investment of $e_i = \phi_x^{-1}(\frac{1+c}{2}) + a_i$.

$$\eta^*_{pTM}(a_i) = c\phi_x^{-1}(\frac{1+c}{2}) + ca_i + \psi_x^{-1}(1/2) \tag{4.9}$$

Note that unlike with women and nontraditional men, this investment level does not depend on the distributions of agents–except to the extent that the distributions satisfy MARGb–and maximizes total surplus if and only if c=1.

We can also confirm that an equilibrium exists:

Lemma 15. *An equilibrium exists.*

Proof. The strategy space is compact by construction and Lemma 20 ensures each agent's utility function is continuous. Therefore, a fixed point exists. □

Now we can establish some properties of equilibria with traditional men. First, we'll show that the assignment where every traditional man gets their ideal partner and everyone invests at the level $e_i = \phi_{e_i}^{-1}(a_i)$ is surplus maximizing. This provides us an efficient educational investment level against which to compare the investments made in equilibrium.

Lemma 16. *Given that Lemma 14 holds, maximizing $e_i - \phi(e_i)/a_i$ for each agent and assigning every traditional man their ideal partner is sufficient to maximize total market surplus.*

Proof. Note that the surplus with traditional men is equal to the surplus in the analogous vertical model, plus $z - \psi(\eta_{TM}(e_i) - c(e_i + \epsilon_i) + \delta)$ for each traditional man. Lemma 12 establishes that the former surplus is maximized by each agent maximizing $e_i - \phi(e_i)/a_i$, regardless of assignment, and the latter component of surplus is maximized when each traditional man receives 0 from the ψ component since $\psi \geq 0$. This occurs when $\eta_{TM} = c(e_i + \epsilon_i) - \delta$, and by Equation 4.10 such an assignment is feasible. □

Next, traditional men always get their ideal type, so their investment behavior doesn't depend on the distributions so long as Lemma 14 is satisfied. Because $c \leq 1$, traditional men must weakly underinvest in education. Note that uniqueness results of Section 3 only apply to finitie markets, so we'll have to assume here that the assignment is unique.

Corollary 17. *Suppose the assignment given by Proposition 6 is unique. Then in any equilibrium satisfying Lemma 14, every traditional man gets their ideal partner and chooses $e \leq e^*$, where e^* is total surplus maximizing.*

Proof. By Proposition 6, every traditional man gets their ideal match, given Equation #.# holds. $e_i = \phi_x^{-1}(\frac{1+c}{2}) + a_i \leq \phi_x^{-1}(1) + a_i = e^*$. □

While traditional men ignore the distributions when choosing effort, women and nontraditional men typically do not. Thus, we'll get inefficient investments if η' isn't equal to 1 in equilibrium. Because there is a higher density of high ability women than high ability nontraditional men–and traditional men don't want to match to high ability women–high ability men are in shortage. Generally, that means high ability women will have to dig deeper into the roster of nontraditional men, matching to men with a wider distribution of abilities than they have. This will generally correspond to a wider distribution of education levels, which will mean that a small increase in a woman's education level yields a large increase in her partner's education level ($\eta'_W > 1$), incentivizing women to overinvest in education to get to the top of the heap while nontraditional men have too little incentive to invest. However, our environment is quite general and rich, so under very special circumstances it may be possible to reverse this relationship, yielding $\eta'_W < 1$ and $\eta'_{NM} > 1$. We'll now characterize the distinctive constraints on education distributions and matching patterns necessary to get these paradoxical exceptions. In particular, when all high type women underinvest, the density of high

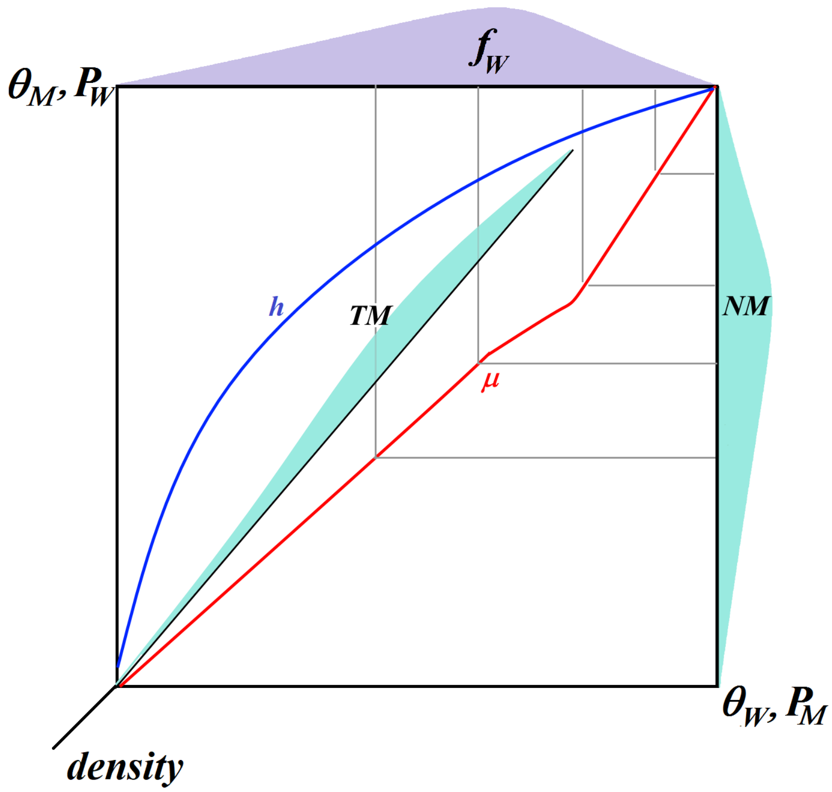

FIGURE 4.1. Illustration of the second stage matching problem. There are fewer nontraditional men than there are women, so high type men are in shortage, and high type women are in surplus. Traditional men match to their own ideal.

education women must be lower than that of the men they match to, but the relative density of women to men must be increasing in education ($\eta''_{NM} < 0$). This allows nontraditional men to dramatically overinvest at the lower levels of ability due to the steeply increasing η while higher type men invest less as η' is lower at higher levels. The net effect is to compress the educational distribution of men while stretching the distribution women, despite the distribution of ability being more compressed for women.

Proposition 18. *There exists a measurable interval of nontraditional men (women) who underinvest (overinvest) in education.*

Proof. Lemma 25, the continuous differentiability of $E[\eta_M](E[\eta_W])$, and the Mean Value Theorem ensure there exists $a \in [a_M, \overline{a}](a \in [a_W, \overline{a}])$ such that $E[\eta_{NM}(e_{NM}(a))] < 1(E[\eta_W(e_W(a))] > 1)$. The continuity of $E[\eta_M](E[\eta_W])$ and $e_M(a)(e_W(a))$ ensure that there exists an open ball about a such that, for all $a' \in B(a)$, $E[\eta_M^{'}(e(a'))] < 1(E[\eta_W^{'}(e(a'))] > 1)$. Agents optimize by choosing effort such that $\frac{1+\frac{\partial}{\partial e}E[\eta_\Gamma(e(a'))]}{2} = \phi_e(e(a') - a) < (>)1 = \phi_e(e^* - a)$ for the efficient effort e^*, so $\phi_{ee} > 0$ ensures that $e(a') < (>)e^*$. □

5. Conclusions

In this paper, we've established an assignment rule for NTU matching problems with single-peaked prefences over a single variable. This rule is amenable to closed form matching functions, and subsumes simpler cases like horizontal and vertical preferences both in results and interpretation. As in Becker style vertical matching, many agents match assortatively in type, via either PAM or NAM. As in matching with horizontal types, the overlap of the two sides' distributions match to their mutually ideal partner, via CPAM. Most interestingly, we find a third, novel regime–agents in shortage are able to leverage this scarcity to obtain matches that they find ideal, but that are not ideal for their partner. Shortage agents exhibit CPAM, while surplus agents exhibit PAM. We've also generalized the SPC for uniqueness of stable assignments, and single-peaked preferences are just one potential application. The dramatic relaxation of assumption MARGb when compared to MARGa, corresponding to the SPC, illustrate the significant degree of generalization. The These results can potentially be applied to a variety of settings, such as job search in fields where salary is non-negotiable, but we've focused on an application to heterosexual marriage. In our application to marriage matching with premarket investments, we've shown how our results can make models with non-vertical preferences tractable, and found some interesting basic results about investment efficiency when some men prefer lower success partners. To wit, the shortage of high success men who prefer high success women causes high ability women overinvest in education in the fierce competition for the few attractive non-traditional men, while high ability non-traditional men underinvest due to the glut of attractive partners. Traditional men engage with the market in an entirely different way, since, being in shortage, they can always marry their ideal partner. This means that changes to the distributions have no effect on their behavior, so long as there are enough women to satisfy feasibility.

Moving forward, there are several potential avenues for further research. The generalized SPC we've developed was motivated by the single-peaked preferences problem posed in this paper, but is far more general. This result may illuminate similarly tractable assignment rules in more complex settings, perhaps those with multi-dimensional preferences. Additionally, the assignment rule provides empirical predictions that can be tested, particularly witth a data set that includes agent preferences. Finally, the results of our illustrative application to matching with premarket investments are novel but extremely basic, and suggest that a more sophisticated analysis may recover deeper insights on investment behavior when preferences are more complex than those typically assumed in the literature.

6. Appendix

Appendix A: Uniqueness. Proposition 9:

Proof. Existence: We'll show that matching each agent to their index match is stable. Clearly, no blocking pair can involve mutually ideal partners. Suppose a blocking pair exists: $w_i m_j$. Without loss, assume $i < j$. Then $w_i \underset{m_j}{\succ} w_j$ and $m_j \underset{w_i}{\succ} m_i$. But then $w_j \underset{m_j}{\succ} w_i$, contradiction.

Uniqueness: Given LEX, mutually ideal partners strictly prefer one another to any other match, and thus cannot be matched to any othe partner stably. Now consider the lowest ordered woman with a different match under an alternative assignment M, w_{k_1}. Stability ensures that $\mu_M(w_{k_1}) \underset{w_{k_1}}{\succ} m_{k_1}$ or $\mu_M(m_{k_1}) \underset{m_{k_1}}{\succ} m_{k_1}$. Without loss, assume the former and define $m_{k_{i+1}} \equiv \mu_M(w_{k_i})$ and $w_{k_i} \equiv \mu_L(m_{k_i})$. By the minimality of k_1 and ORD1, we have $w_{k_2} \underset{m_{k_2}}{\succ} w_{k_1}$. But since we have $w_{k_1} m_{k_2}$, there must be m_{k_3} such that $w_{k_2} m_{k_3}$. so $m_{k_3} \underset{w_{k_2}}{\succ} m_{k_2}$. Now suppose $m_{k_{m+1}} \underset{w_{k_m}}{\succ} m_{k_m}$ and $w_{k_m} m_{k_{m+1}}$ for all $m < n$. If $k_n > k_{n-1}$, ORD1 implies $w_{k_n} \underset{m_{k_n}}{\succ} w_{k_{n-1}}$, and this along with $w_{k_n}m_{k_{n+1}}$ implies $m_{k_{n+1}} \underset{w_{k_n}}{\succ} m_{k_n}$. If $k_n < k_{n-1}$, suppose $w_{k_{n-1}} \underset{m_{k_n}}{\succ} w_{k_n}$. Then $m_{k_{n-1}} \underset{w_{k_{n-1}}}{\succ} m_{k_n}$. Contradiction. Then $w_{k_n} \underset{m_{k_n}}{\succ} w_{k_{n-1}}$, and this along with $w_{k_n}m_{k_{n+1}}$ implies $m_{k_{n+1}} \underset{w_{k_n}}{\succ} m_{k_n}$. Then by Lemma 8, we must reach $w_{k_N}m_{k_1}$, with $m_{k_1} \underset{w_{k_N}}{\succ} m_{k_N}$. But ORD2 implies that $m_{k_N} \underset{w_{k_N}}{\succ} m_{k_1}$. Contradiction. □

Proposition 10:

Proof. Given the order induced by the assignment order in the construction of L, we'll show that ORD holds. Suppose $j > i$ and $m_i \underset{w_i}{\prec} m_j$. By the construction of L, $P_{w_i} > \theta_{m_i}$. If $m_i \underset{w_i}{\prec} m_j$, then $\theta_{m_j} > \theta_{m_i}$ and the monotonicity of g ensures that either $P_{m_j} = \theta_{w_j}$ or $\theta_{m_j} < \theta_{m_i}$, so $P_{m_j} = \theta_{w_j} < \theta_{w_i}$. This implies $w_j \underset{m_j}{\succ} w_i$. Now we must show ORD2. $m_{k_{N+1}} \underset{w_{k_N}}{\succ} m_{k_N}$ and $m_{k_N} \underset{w_{k_N}}{\succ} m_{k_i}$ imply $m_{k_{N+1}} \underset{w_{k_N}}{\succ} m_{k_i}$ and the latter is given, so we need to show that $m_{k_N} \underset{w_{k_N}}{\succ} m_{k_i}$. $P_{w_{k_N}} > \theta_{m_{k_N}}$, so it will suffice to show $\theta_{m_{k_N}} > \theta_{m_i}$. $m_{k_2} \underset{w_i}{\succ} m_i$ implies $P_{w_i} > \theta_{m_i}$ and thus $\theta_{m_{k_2}} > \theta_{m_i}$. Now suppose $m_{k_{n+1}} \underset{w_{k_n}}{\succ} m_{k_n}$ and $\theta_{m_{k_n}} > \theta_{m_i}$. Then $P_{w_{k_n}} > \theta_{m_{k_n}}$, so $\theta_{m_{k_{n+1}}} > \theta_{m_{k_n}} > \theta_{m_i}$. □

Appendix B: Equilibrium in the Marriage Market with Investments. Need to fix T's below this point.

Define a_W as the lowest type for who never matches to a traditional man: $e_W(a_W) - \overline{\epsilon} > \psi'^{-1}\left[\frac{1}{2}\right] + c(\overline{a} + \phi_e^{-1}\left(\frac{2}{c+1}\right)) + c\overline{\epsilon}$. $\overline{a} - \overline{\epsilon} > \psi'^{-1}\left[\frac{1}{2}\right] + c(\overline{a} + \phi_e^{-1}\left(\frac{2}{c+1}\right)) + c\overline{\epsilon}$ so $a_W < \overline{a}$ exists. Define a_M as the ability of the nontraditional man for whom $\int_{e_W(a_W).}^{\overline{\theta}_w} f_W(\theta)\,d\theta = \int_{e_{NM}(a_M)}^{\overline{\theta}_m} f_{NM}(\theta)E_\theta(1-T(a))\,d\theta$.

Lemma 14:

Proof. i. Feasibility requires

$$\int_{x}^{\infty} f_W(\theta)d\theta a = \int_{\frac{x-\psi_x^{-1}(1/2)+\delta}{c}}^{\infty} E_\theta T(a) f_{TM}(\theta)d\theta + \int_{g(x)}^{\infty} E_\theta(1-T(a))f_{NM}(\theta)d\theta$$

need to show that

$$\int_{x}^{\infty} f_W(\theta)d\theta > \int_{\frac{x-\psi_x^{-1}(1/2)+\delta}{c}}^{\infty} E_\theta T(a) f_{TM}(\theta)d\theta + \int_{x}^{\infty} E_\theta(1-T(a))f_{NM}(\theta)d\theta$$

$\int_{g(x)}^{\infty} E_\theta(1-T(a))f_{NM}(\theta)d\theta$ is decreasing in the lower limit of integration, so this implies $g(x) < x$.

Suppose $\overline{a}(1-c) - \psi_x^{-1}(1/2) + \delta - 2\overline{\epsilon} > \overline{\epsilon}C/(\underline{T}\xi)$. Suppose $\delta \geq \psi_x^{-1}(1/2)$ and $c < 1$. Then there exists $y > 0$ such that, if $max\{\overline{e}, \overline{\epsilon}\} < y$, the inequality must hold. It suffices to show that

$$\int_{x+\overline{e}+\overline{\epsilon}}^{\infty} h(t)dt > \int_{\frac{x-\overline{\epsilon}-\psi_x^{-1}(1/2)+\delta}{c}}^{\infty} T(t)h(t)dt + \int_{x-\overline{\epsilon}}^{\infty} (1-T(t))h(t)dt$$

Which we can rewrite as

$$\int_{x+\overline{e}+\overline{\epsilon}}^{\frac{x-\overline{\epsilon}-\psi_x^{-1}(1/2)+\delta}{c}} h(t)dt \geq \int_{x-\overline{\epsilon}}^{\frac{x-\overline{\epsilon}-\psi_x^{-1}(1/2)+\delta}{c}} (1-T(t))h(t)dt$$

This is satisfied if

$$\int_{x+\overline{\epsilon}}^{\frac{x-\overline{\epsilon}-\psi_x^{-1}(1/2)+\delta}{c}} h(t)dt - \overline{e}C \geq (1-\underline{T}) \int_{x-\overline{\epsilon}}^{\frac{x-\overline{\epsilon}-\psi_x^{-1}(1/2)+\delta}{c}} h(t)dt$$

That is,

$$\int_{x+\overline{\epsilon}}^{\frac{x-\overline{\epsilon}-\psi_x^{-1}(1/2)+\delta}{c}} h(t)dt \geq \overline{e}C/\underline{T}$$

There are two cases: a) $\frac{x-\psi_x^{-1}(1/2)+\delta+\overline{\epsilon}}{c} > \overline{a}$, where even the highest type traditional man prefers a partner lower than x, and b) $\frac{x-\psi_x^{-1}(1/2)+\delta+\overline{\epsilon}}{c} \leq \overline{a}$. For a)

$$\int_{x+\overline{\epsilon}}^{\overline{a}} h(t)dt \geq \overline{e}C/\underline{T}$$

Which holds if

$$\overline{a} - (x+\overline{\epsilon}) \geq \overline{e}C/(\xi\underline{T})$$

Which holds if

$$\overline{a}(1-c) - \psi_x^{-1}(1/2) + \delta \geq \overline{e}C/(\underline{T}\xi).$$

For b),

$$(1-c)/c * x + \frac{\delta-(1+c)\overline{\epsilon}-\psi_x^{-1}(1/2)}{c} \geq \overline{e}C/(\xi\underline{T})$$

which holds if

$$\overline{a}(1-c) - \psi_x^{-1}(1/2) + \delta - 2\overline{\epsilon} \geq \overline{e}C/(\xi\underline{T}).$$

ii):

Want to show $cf_W(\theta) \geq E[T(\frac{\theta-\psi_x^{-1}(1/2)+\delta}{c})]f_{TM}(\frac{\theta-\psi_x^{-1}(1/2)+\delta}{c})$ for all $\theta \in \Theta$.

$\frac{c\xi}{2} \geq E[T(\frac{\theta-\psi_x^{-1}(1/2)+\delta}{c})]f_{TM}(\frac{\theta-\psi_x^{-1}(1/2)+\delta}{c})$.

We can choose T such that

$\underline{T}C \geq E[T(\frac{\theta-\psi_x^{-1}(1/2)+\delta}{c})]f_{TM}(\frac{\theta-\psi_x^{-1}(1/2)+\delta}{c})$.

For sufficiently small $\underline{T}$, $\frac{c\xi}{2} \geq \underline{T}C$. Therefore $cf_W(\theta) \geq E[T(\frac{\theta-\psi_x^{-1}(1/2)+\delta}{c})]f_{TM}(\frac{\theta-\psi_x^{-1}(1/2)+\delta}{c})$ and Lemma 4 ensures MARGb holds. □

Lemma 19. *$a_m \le a_w - (\overline{a} - a_w - 2\overline{\epsilon} - 2\overline{e})\xi\overline{T}/(C(1-\overline{T})) + 4\overline{\epsilon} + 4\overline{e}$. Additionally, $\int_{\theta_w}^{\overline{\theta}_w} f_W(\theta)\,d\theta = \int_{\theta_m}^{\overline{\theta}_m} f_{NM}(\theta)E_\theta(1-T(a))\,d\theta$ implies $\theta_m \le a_w - (\overline{a} - a_w - 2\overline{\epsilon} - 2\overline{e})\xi\overline{T}/(C(1-\overline{T})) + 3\overline{\epsilon} + 3\overline{e}$.*

Proof. Suppose $\int_{\theta_w}^{\overline{\theta}_w} f_W(\theta)\,d\theta = \int_{\theta_m}^{\overline{\theta}_m} f_{NM}(\theta)E_\theta(1-T(a))\,d\theta$. Then $\int_{a_w+2\overline{\epsilon}+2\overline{e}}^{\overline{a}} h(a)\,da \le \int_{\theta_m-\overline{\epsilon}-\overline{e}}^{\overline{a}} h(a)(1-T(a))\,da$. Then $\int_{a_w+2\overline{\epsilon}+2\overline{e}}^{\overline{a}} h(a)\,da \le (1-\overline{T})\int_{\theta_m-\overline{\epsilon}-\overline{e}}^{\overline{a}} h(a)\,da$. Then $\overline{T}\int_{a_w+2\overline{\epsilon}+2\overline{e}}^{\overline{a}} h(a)\,da \le (1-\overline{T})\int_{\theta_m-\overline{\epsilon}-\overline{e}}^{a_w+2\overline{\epsilon}+2\overline{e}} h(a)\,da$. Then $\overline{T}(\overline{a} - a_w - 2\overline{\epsilon} - 2\overline{e})\xi \le (1-\overline{T}) * C(a_w + 2\overline{\epsilon} + 2\overline{e} - (\theta_m - \overline{\epsilon} - \overline{e}))$. Then $(\overline{a} - a_w - 2\overline{\epsilon} - 2\overline{e})\xi \le (1-\overline{T})/\overline{T} * C(a_w + 3\overline{\epsilon} + 3\overline{e} - \theta_m)$. Then $(\overline{a} - a_w - 2\overline{\epsilon} - 2\overline{e})\xi\overline{T}/(C(1-\overline{T})) - 3\overline{\epsilon} - 3\overline{e} \le a_w - \theta_m$. Then $\theta_m \le a_w - (\overline{a} - a_w - 2\overline{\epsilon} - 2\overline{e})\xi\overline{T}/(C(1-\overline{T})) + 3\overline{\epsilon} + 3\overline{e}$. Given that $\theta_m \le a_m + \overline{\epsilon} + \overline{e}$. It also follows that $a_m \le a_w - (\overline{a} - a_w - 2\overline{\epsilon} - 2\overline{e})\xi\overline{T}/(C(1-\overline{T})) + 4\overline{\epsilon} + 4\overline{e}$. □

Lemma 20. *$\frac{\partial}{\partial e}E[\eta_\Gamma(e)]$ and $\frac{\partial^2}{\partial e^2}E[\eta_\Gamma(e)]$ are well defined for $\Gamma \in \{W, NM, TM\}$. $|\frac{\partial}{\partial e}E[\eta_\Gamma(e)]/$ and $|\frac{\partial^2}{\partial e^2}E[\eta_\Gamma(e)]|$ are bounded above.*

Proof. $\frac{\partial}{\partial e}E[\eta_\Gamma(e)] = \int_{e-\overline{\epsilon}}^{e+\overline{\epsilon}} E[\theta_p(x)]q'(x-e)\,dx$, which is well defined as q is twice differentiable. Similarly, $\frac{\partial^2}{\partial e^2}E[\eta_\Gamma(e+\epsilon)] = \int_{e-\xi}^{e+\xi} E[\theta_p(x)]q''(x-e)\,dx$, which is well defined as q is twice differentiable. $E[\theta_p(x)] \in [0, \bar{a} + \overline{e} + \overline{\epsilon}]$ for all x, and since q and q' are Lipschitz continuous, $|\int_{e-\overline{\epsilon}}^{e+\overline{\epsilon}} E[\theta_p(x)]q'(x-e)\,dx| \le |\int_{e-\overline{\epsilon}}^{e+\overline{\epsilon}} (\bar{a} + \overline{e} + \overline{\epsilon})k_1\,dx| = (\bar{a} + \overline{e} + \overline{\epsilon})\xi k_1$ and $|\int_{e-\overline{\epsilon}}^{e+\overline{\epsilon}} E[\theta_p(x)]q''(x-e)\,dx| \le |\int_{e-\overline{\epsilon}}^{e+\overline{\epsilon}} (\bar{a} + \overline{e} + \overline{\epsilon})k_2\,dx| = (\bar{a} + \overline{e} + \overline{\epsilon})\xi k_2$ □

Lemma 21. *Each agent has a unique optimal effort level $e_i^*(\sigma)$ given the strategy profile.*

Proof. Traditional men have optimal effort levels given by $\phi_x^{-1}(\frac{1+c}{2}) + a_i$. All other agents maximize $\frac{e_i + E[\eta_\Gamma(e_i)]}{2} - \phi(e_i - a)$, which is twice continuously differentiable by Lemma 20. Also by By Lemma 20 , $\frac{\partial^2}{\partial e^2}(\frac{E[\eta_\Gamma(e_i)]}{2} - \phi(e_i - a_i)) = \frac{\frac{\partial^2}{\partial e^2}E[\eta_\Gamma(e_i)]}{2} - \phi''(e_i - a_i) \le \frac{(\bar{a}+\xi)\xi k_1}{2} - \phi''(e_i) < 0$, so expected utility is strictly concave. Additionally, for any a there exists e_a such that $(\bar{a} + \overline{e} + \overline{\epsilon})\xi k_1 < \phi'(e_a - a)$. Then the optimal effort must be an element of $[0, e_a]$, a compact interval. Thus, utility achieves a unique maximum in e_i. □

Lemma 22. *The optimal e_i is a continous function of a.*

Proof. $E[a] \equiv [a, \overline{e}]$ is continuous in a and compact-valued. Then by the Maximum Theorem, $e_i(a)$ is upper hemicontinuous in a, and, since it is a well defined function by Lemma 21, is also continuous. Can also show $a + e(a)$ strictly increasing: Expected match quality continuous in e and costs of match quality decreasing in a. □

Corollary 23. *$E[\theta]$ has full support on $[a_\Gamma + \overline{e}, \overline{a}]$.*

Proof. h_S has full support on $[a_\Gamma, \overline{a}]$, $a + e_i[a]$ is continuous in a, and $e_i[\overline{a}] > 0$, $e_i[a] < \overline{e}$. □

Lemma 24. *Suppose $q(-x) = q(x)$, $q(|x|) \le q(|y|)$ if $|y| \le |x|$, and $Q(\overline{e}/2) - Q(-\overline{e}/2) \le 1/2$ for all $x, y \in [-\overline{\epsilon}, \overline{\epsilon}]$. Then $f_W(\theta) > \frac{\xi}{2}$.*

Proof. $h(a) > \xi$ for all $a \in [\underline{a}, \overline{a}]$, so for a lower bound we'll assume $h = \xi$. To minimize the density $f_W(\theta)$ agents must choose an e that minimizes $q(e - \theta)$. Thus, agents of ability $a < \theta - \overline{e}/2$ choose $e = a$ and agents of ability $a \ge \theta - \overline{e}/2$ choose $e = a + \overline{e}$. Then $f_W(\theta) = \int_{\theta-\overline{\epsilon}}^{\theta-\overline{e}/2} h(x)q(\theta - x)\,dx + \int_{\theta+\overline{e}/2}^{\theta+\overline{\epsilon}} h(x)q(\theta - x)\,dx \ge \xi(\int_{\theta-\overline{\epsilon}}^{\theta-\overline{e}/2} q(x-\theta)\,dx + \int_{\theta+\overline{e}/2}^{\theta+\overline{\epsilon}} q(x-\theta)\,dx) \ge \xi/2$. □

Lemma 25. *There exists $y > 0$ such that, if $max\{\overline{\epsilon}, \overline{e}\} < y$, (a) $E[\eta_{NM}(e_{NM}(\overline{a}))] - E[\eta_{NM}(e_{NM}(a_M))] < e_{NM}(\overline{a}) - e_{NM}(a_M)$ and (b) $E[\eta_W(e_W(\overline{a}))] - E[\eta_W(e_W(a_W))] > e_W(\overline{a}) - e_W(a_W)$.*

Proof. (a) $(\overline{a}-a_W-2\overline{\epsilon}-2\overline{e})\xi\overline{T}/(C(1-\overline{T}))-6\overline{\epsilon}-7\overline{e}>0$ for sufficiently small y. Since $E[\eta_{NM}(e(\overline{a}))]\leq\overline{a}+\overline{e}+\overline{\epsilon}$. $E[\eta_{NM}(e(a_M))]\geq a_W-\overline{e}-\overline{\epsilon}$, Lemma 19 guarantees $a_M\leq a_W-2\overline{\epsilon}-3\overline{e}$ and $E[\eta_M(e(\overline{a}))]-E[\eta_M(e(a_M))]\leq \overline{a}-a_W+2\overline{e}+2\overline{\epsilon}<\overline{a}-a_M-\overline{e}<e(\overline{a})-e(a_M)$.

(b) Corollary to Lemma 5, $\eta_{pW}(\theta)$ is increasing in θ, so $E(\eta_W(y_W(\overline{a}))\geq\eta_{pW}(\theta_w)\equiv\theta_m$ where $e_W(\overline{a})\geq\theta_W+\overline{\epsilon}$. This is satisfied by $\theta_W=\overline{a}-\overline{\epsilon}$. To find partner success θ_m, we must have $\int_{\theta_w}^{\overline{\theta}_w} f_W(\theta)\,d\theta=\int_{\theta_m}^{\overline{\theta}_m} f_{NM}(\theta)E_\theta(1-T(a))\,d\theta$. Then $(\overline{\theta}_w-\theta_w)C\geq(1-\overline{T})\xi(\overline{\theta}_m-\theta_m)$ Thus $E(\eta_W(e_W(\overline{a}))\geq\theta_m\geq\overline{a}-2\overline{\epsilon}-(\overline{e}+\overline{2\epsilon})\frac{C}{(1-\overline{T})\xi}$. Similarly, $E(\eta_w(e_w(a_w))\leq\eta_{pw}(\theta_w)\equiv\theta_m$ where $e_w(a_w)\leq\theta_w-\overline{\epsilon}-\overline{e}$. This is satisfied by $\theta_w=a_w+\overline{\epsilon}+\overline{e}$. To find partner success θ_m, we must have $\int_{\theta_w}^{\overline{\theta}_w} f_W(\theta)\,d\theta=\int_{\theta_m}^{\overline{\theta}_m} f_{NM}(\theta)E_\theta(1-T(a))\,d\theta$ and $\int_{a_w+2\overline{\epsilon}+2\overline{e}}^{\overline{a}} h(a)\,da\leq(1-\underline{T})\int_{\theta_m-\overline{\epsilon}-\overline{e}}^{\overline{a}} h(a)\,da$. Then $\underline{T}\int_{a_w+2\overline{\epsilon}+2\overline{e}}^{\overline{a}} h(a)\,da\leq(1-\underline{T})\int_{\theta_m-\overline{\epsilon}-\overline{e}}^{a_w+2\overline{\epsilon}+2\overline{e}} h(a)\,da$ and $(\overline{a}-a_w-2\overline{\epsilon}-2\overline{e})\xi\leq(1-\underline{T})/\underline{T}*C(a_w+2\overline{\epsilon}+2\overline{e}-(\theta_m-\overline{\epsilon}-\overline{e}))$ Thus $E[\eta_W(e_W(a_W))]\leq\theta_m\leq a_w+3\overline{\epsilon}+3\overline{e}-(\overline{a}-a_w-2\overline{\epsilon}-2\overline{e})\frac{\xi\underline{T}}{C(1-\underline{T})}$. Then, for positive constants c_ϵ and c_e, we have $E(\eta_w(e_w(\overline{a}))-E(\eta_w(e_w(a_w))\geq \overline{a}-2\overline{\epsilon}-(\overline{e}+\overline{2\epsilon})\frac{C}{(1-\overline{T})\xi}-(a_w+3\overline{\epsilon}+3\overline{e}-(\overline{a}-a_w-2\overline{\epsilon}-2\overline{e})\frac{\xi\underline{T}}{C(1-\underline{T})})=(\overline{a}-a_w)(1+\frac{\xi\underline{T}}{C(1-\underline{T})})-c_\epsilon\overline{\epsilon}-c_e\overline{e}> \overline{a}-a_W+\overline{e}\geq e_W(\overline{a})-e_W(a_W)$ for y sufficiently small. □

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